PSEUDO-EINSTEIN AND Q-FLAT METRICS WITH EIGENVALUE ESTIMATES ON CR-HYPERSURFACES

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Abstract.

In this paper, we will use the Kohn's $\bar{\partial}_b$-theory on CR-hypersurfaces to derive some new results in CR-geometry.

Main Theorem. *Let M^{2n-1} be the smooth boundary of a bounded strongly pseudo-convex domain Ω in a complete Stein manifold V^{2n}. Then (1) For $n \geq 3$, M^{2n-1} admits a pseudo-Einstein metric; (2) For $n \geq 2$, M^{2n-1} admits a Fefferman metric of zero CR Q-curvature; and (3) for a compact strictly pseudoconvex CR emendable 3-manifold M^3, its CR Paneitz operator P is a closed operator.*

There are examples of non-emendable strongly pseudoconvex CR manifold M^3, for which the corresponding $\bar{\partial}_b$-operator and Paneitz operators are not closed operators.

0. Introduction

In this paper, we study several questions, including the existence of Q-flat metrics, pseudo-Einstein metrics and the closedness of the CR Paneitz operators.

First, we will use an approach proposed by Fefferman and his school to prove that "the complete Kähler-Einstein g_∞ on an open domain Ω induces a metric on $M = b\Omega$ with zero CR Q-curvature, where Ω is a smooth, bounded strictly pseudo-convex domain in a Stein manifold V^{2n}." To achieve this goal, we solve a $\partial\bar{\partial}$-Poincaré-LeLong equation via the $\bar{\partial}$-theory. Although this part does not produce new hard a-priori estimates, it is still valuable for other potential applications.

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The second purpose is to prove the existence of pseudo-Einstein metrics on strictly pseudo-convex CR-hypersurface of real dimension ≥ 5 through solving the $\bar{\partial}_b$ Poincaré-LeLong equations.

The last part of our paper is to study the closedness of CR Paneitz operator, which is a fourth-order differential operator. It is known that the positivity of CR Paneitz operator is related to the deformation of Q-curvatures under the conformal change of metrics on Riemannian manifold M^m. In particular, the positivity of CR Paneitz operator is also related to the lower bound of the first eigenvalue of sub-Laplace on a CR manifold M^3, see [CC], [CCC] and [LL]. It will be shown that, if $M^3 = b\Omega^4$ is the smooth boundary of bounded strictly pseudo-convex domain Ω in a Stein manifold V^4, then its CR-Paneitz operator on M^3 is closed, for any metric on M^3.

Main Theorem. *Let M^{2n-1} be the smooth boundary of a bounded strongly pseudo-convex domain Ω in a complete Stein manifold V^{2n}. Then*

(1) For $n \geq 2$, M^{2n-1} admits a metric of zero CR Q-curvature;

(2) For $n \geq 3$, M^{2n-1} admits a pseudo-Einstein metric;

(3) In addition, for a compact strictly pseudoconvex CR emendable 3-manifold M^3, its CR Paneitz operator P is a closed operator.

Earlier work in this direction for the case of $V^{2n} = \mathbb{C}^n$ can be found in [L2], [FH] and [GG]. In a very recent paper [LL], Li and Luk obtained an explicit formula for Webster's pseudo-Ricci curvature on real hypersurfaces in $\mathbb{C}^n$. Thus, their result could lead another proof of Cheng-Yau's result ([CY]) and Mok-Yau's theorem [MY], which will be used in Section 2 below.

Among other things, we introduce some new methods to handle pseudo-Einstein metric and Paneitz operators in this paper. For example, we use the closeness of $\bar{\partial}_b$ and $\bar{\partial}_b^*$ operators provided by Kohn's theory, in order to complete the proof. When $\dim_{\mathbb{R}}[M] = 3$, we decompose the Paneitz operator P as a product of closed

operators. Thus, the closed property of P will follow immediately, see Lemma 1.4 and Section 4 below.

1. Preliminary results

It is well-known that the real Laplace $\triangle$ on a Kähler manifold V^{2n} satisfies

$$\triangle = 2\Box = 2\bar{\Box},$$

where $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ is the complex Laplace operator. However, it may happen that $\triangle_b \neq 2\Box_b$ in some cases. Let us recall the notions of $\triangle_b$ and $\Box_b$.

Since $M^{2n-1} = bV^{2n}$ has odd real dimension, it is a Cauchy-Riemann manifold. The $\bar{\partial}_b$ operator induces a sub-elliptic operator

$$\Box_b = \bar{\partial}_b^*\bar{\partial}_b + \bar{\partial}_b\bar{\partial}_b^*$$

acting on $L^2_{(p,q)}(M)$. Similarly, there is a real sub-Laplace operator, which can be viewed as partial trace of the hessian operator (or can be viewed a sum of the squares of $(2n-2)$ vectors):

$$\triangle_b u|_z = \sum_{k=1}^{2n-2} \langle \nabla_{e_k}(\nabla^b u), e_k\rangle|_z$$

where e_{2n} is the outward real unit normal vector of Ω along boundary $M = b\Omega$, $e_{2j} = Je_{2j-1}$ for $j = 1, ..., n$, $z \in M$, J is the complex structure of V^{2n}, $\{e_1, e_2, \cdots, e_{2n-3}, e_{2n-2}, e_{2n-1}, e_{2n}\}$ is an orthonormal basis of $[T_z(V)]_{\mathbb{R}}$ and

$$\nabla^b u = \sum_{k=1}^{2n-2} du(e_k)e_k.$$

When Ω has the strongly pseudo-convex boundary in a Stein manifold V^{2n} with $n = 2$, it has been observed that

$$\Box_b u = \frac{1}{2}[\triangle_b u + \sqrt{-1}Tu] \tag{1.1}$$

for all $u \in L^2(M^3)$, where $T = \lambda e_3$ is the Reeb vector of the CR 3-manifold M^3 for some real valued function λ, see [L1, p414].

The operator $\Box_b$ is a Lewy type operator, which may *not* be locally solvable.

If the Reeb vector T induces an infinitesimal pseudo-conformal with respect to the Tanaka-Webster metric, then the torsion of M^3 is zero, see [Web, p33]. In this case, the operator $\Box_b$ is related to the so-called CR Paneitz operator P, where P is given by

$$Pu = \triangle_b^2 u + T^2 u = 4\Box_b \bar{\Box}_b u, \tag{1.2}$$

for $u \in L^2(M^3)$. More generally, if M^3 has torsion free in the sense of Tanaka (cf. [Ta1-2] [Web]), then (1.2) holds.

The eigenvalues of the Paneitz operator and CR Paneitz operators have been considered various authors ([Ch], [CC]). The eigenvalue estimate plays an important role to the study of the so-called Q-curvature flow, see [Br] [CCC].

Definition 1.1. *(1) The CR-Paneitz operator $P : L^2(M^3) \to L^2(M^3)$ is called essentially positive, if there is a positive constant $\lambda_1 > 0$ such that*

$$\langle Pu, u\rangle\| \geq \lambda_1 \|u\|^2, \tag{1.3}$$

for all $u \perp ker(P)$.

(2) The operator $\mathcal{F} : L^2_{(p,q)}(M) \to L^2_{(p,q)}(M)$ is said to have positive spectrum gap at 0 (or is said to be a closed operator) if there is a positive constant $\lambda_{p,q} > 0$ such that

$$\|\mathcal{F}u\| \geq \lambda_{p,q}\|u\|, \tag{1.4}$$

for all $u \perp [L^2_{(p,q)}(M) \cap ker(\mathcal{F})]$.

(3) A smooth function $f : U_\varepsilon(M) \to \mathbb{R}$ is called a defining function of M if $f^{-1}(0) = M$ and if 0 is not a critical value of f, where $U_\varepsilon(M) \subsetneq V^{2n}$ is a neighborhood of M in a Stein manifold V^{2n}.

(4) Let θ be a contact 1-form of M^{2n-1} and $J : \ker\theta \to \ker\theta$ be the almost complex structure on the CR-distribution $\ker\theta$ such that $J^2\vec{v} = -\vec{v}$ for all $\vec{v} \in \ker\theta$.

In what follows, we always let

$$[T^{(1,0)}(M) \oplus T^{(0,1)}(M)] = [\ker\theta] \bigotimes_{\mathbb{R}} \mathbb{C}.$$

(5) A CR manifold M^{2n-1} is said to have transverse symmetry or torsion-free if it admits a CR Reeb vector field ξ such that $\xi \notin \ker\theta$ with

$$\mathcal{L}_\xi J = 0$$

where $\mathcal{L}$ is the Lie derivative and J is the complex structure of $[T^{(1,0)}(M) \oplus T^{(0,1)}(M)]$.

If ξ is the real part of a holomorphic vector filed $\tilde{X}$ on a neighborhood $U_\varepsilon(M)$ of M, then ξ induces an automorphism on $U_\varepsilon(M)$. Any real part ξ of a holomorphic vector filed restricted to M induces a CR-automorphism of M.

In the Hörmand-Kohn L^2-theory and the Kohn-Rossi theory, the essential spectrum of $\Box$ and $\Box_b$ have been extensively investigated.

A smooth (p,q)-form u on Ω with $q \geq 1$ is said to satisfy the $\bar{\partial}$-Neumann boundary condition if

$$u((\bar{\partial}\rho)_\#, ...)|_z = 0$$

for all $z \in M = b\Omega$, where $(\bar{\partial}\rho)_\#$ is the complex normal vector field of type $(0,1)$ along the boundary M^{2n-1}.

Theorem 1.2. *([CS], [CaWS]) Let Ω be a bounded domain with smooth pseudoconvex boundary M in a complete Hermitian manifold V^{2n}. Suppose that V^{2n} is either a Stein manifold or $\mathbb{C}P^n$. Then the complex Laplace operator $\Box$ is*

(1) positive for on $L^2_{(p,q)}(\Omega)$ with $(n-1) \geq q \geq 1$; and

(2) essentially positive on $L^2_{(p,0)}(\Omega)$ and $L^2_{(p,n)}(\Omega)$

with respect to $\bar{\partial}$-Neumann boundary condition on $M = b\Omega$.

Moreover, for any Hermitian metric on Ω, the operator $\Box$ is essentially positive on Ω with respect to $\bar{\partial}$-Neumann boundary condition on M.

For the L^2 estimates of $\Box$, the domains Ω in Theorem 1.2 are not necessarily strictly pseudo-convex. However, for estimates of $\Box_b$ on the boundary M^{2n-1} of Ω, we need extra assumptions on M^{2n-1}.

The dual of $\bar{\partial}$-Neumann problem is the so-called $\bar{\partial}$-Cauchy problem. A (p,q)-form u is said to satisfy the Cauchy boundary condition on $M = b\Omega$ if

$$u(\xi, ...)|_z = 0$$

for all $\xi \in T_z^{(0,1)}(M)$ and $z \in M$. If a $\bar{\partial}$-closed form $f \in C^\infty_{(p,q+1)}(\Omega)$ with a compact support in Ω, then one consider to solve $\bar{\partial}u = f$ such that u has a compact support in Ω as well. Solving $\bar{\partial}u = f$ with compact support is related to the $\bar{\partial}$-extension problem, via the Kohn-Rossi theory. Using the solution to the $\bar{\partial}$-extension problem and Theorem 1.2, we are able to solve $\bar{\partial}_b u = f$ on a special class of CR-manifolds:

Theorem 1.3. *([CS], [CaSW]) Let Ω be a bounded Hermitian manifold with a smooth pseudo-convex boundary M. Suppose that one of the following conditions holds:*

(1) Ω is a domain of a complete Stein manifold V^{2n};

(2) $\Omega \subset \mathbb{C}P^n$, and $M = b\Omega$ admits a pluri-subharmonic defining function.

Then the $\bar{\partial}$-Cauchy boundary problem is solvable on Ω. Furthermore, (1) $\bar{\partial}_b$-operator is closed; and (2) the operator $\Box_b : L^2_{(p,q)}(M) \to L^2_{(p,q)}(M)$ is positive for $1 \le q \le n-2$ and essentially positive for $q = 0$ or $q = n-1$.

When $M = b\Omega$ is strongly pseudo-convex, it is well-known that M admits a pluri-subharmonic defining function, see [DF].

If $\mathcal{L} : H_1 \to H_2$ is a linear operator, we let $\mathrm{Dom}(\mathcal{L})$ be its domain and $\mathcal{R}(\mathcal{L})$ be its range. If $A \subset H$ is a subset of a Hilbert space H, the closure of A in H is denoted by $\bar{A}$.

We begin with an elementary but useful criterion for closed operators.

Lemma 1.4. *([CS, p60] or [Hö1-2]) Let $\mathcal{L} : H_1 \to H_2$ be a linear, closed, densely*

defined operator from the Hilbert space H_1 to another Hilbert space H_2. The following conditions on $\mathcal{L}$ are equivalent:

(1) The range $\mathcal{R}(\mathcal{L})$ of $\mathcal{L}$ is closed;

(2) There is a constant C such that

$$\|f\|_1 \leq C\|\mathcal{L}f\|_2$$

for all $f \in Dom(\mathcal{L}) \cap \mathcal{R}(\mathcal{L}^*)$;

(3) The range $\mathcal{R}(\mathcal{L}^*)$ of $\mathcal{L}^*$ is closed;

(4) There is a constant C such that

$$\|f\|_2 \leq C\|\mathcal{L}^*f\|_1$$

for all $f \in Dom(\mathcal{L}^*) \cap \mathcal{R}(\mathcal{L})$.

2. The existence of CR Q-flat metrics on strictly pseudo-convex CR-hypersurfaces in a Stein manifold

In this section, we first recall an existence result of CR Q-flat metrics on CR-hypersurfaces in Euclidean space $\mathbb{C}^n$ due to Fefferman and others. Afterwards, we will extend such a result to CR-hypersurfaces in an arbitrary Stein Manifold V^{2n}. One of our key steps is to use the $\bar{\partial}$-theory to introduce the generalized Fefferman's functional $u \to \hat{J}(u)$, which is independent of the choice of local holomorphic coordinates, see (2.5) below.

2.a. A sufficient condition for existence of Q-flat metrics on real hypersurfaces.

Let us recall a sufficient condition for existence of Q-flat metrics on real hypersurfaces, which were derived by Fefferman and others.

Proposition 2.0. *([FG1-2], [GG]) Let $\Omega \subset \mathbb{C}^n$ be a compact domain with smooth boundary $M^{2n-1} = b\Omega$ in the complex Euclidean space $\mathbb{C}^n$. Suppose Σ^{2n} is an unit*

circle bundle defined on a CR-hypersurface M^{2n-1} and suppose that Σ^{2n} admits an S^1-invariant Einstein-Lorentz metric $g_u^+ = i\partial\bar{\partial}H_u|_{\Sigma^{2n}}$ defined as below. Then M^{2n-1} admits a metric of zero CR Q-metric.

We now provide a description of the metric g_u^+ stated in Proposition 2.0, which will be used for any real hypersurface M^{2n-1} in a Stein manifold V^{2n} as well.

Let K^* be the canonical bundle of V^{2n} restrict to M and let $\Sigma^{2n} = K^*/\mathbb{R}^+$ be the unit circle bundle of K^*. Thus there is a fiberation

$$S^1 \to \Sigma^{2n} \to M^{2n-1}$$

and $\dim_{\mathbb{R}}(\Sigma^{2n}) = 2n$.

We may assume that $\Omega \subset V^{2n}$ is an open strictly pseudo-convex domain with compact smooth boundary $M^{2n-1} = b\Omega$. Suppose that $\hat{u}$ is a defining function of M^{2n-1}. For example, we can choose $\hat{u}$ as a signed distance function form M:

$$\hat{u}(z) = \begin{cases} -d(z, M), & if \quad z \in \Omega \\ d(z, M), & if \quad z \notin \Omega \end{cases}$$

Any other defining function u can be expressed as

$$u(z) = e^{\eta}\hat{u}$$

for some real valued function η.

The contact structure on M is an 1-form given by

$$\theta_u(\xi) = du(J\xi)$$

for all $\xi \in [T(M)]_{\mathbb{R}}$, where J is the complex structure of V^{2n}.

There are two types of metrics which we will use. The first one is the Cheng-Yau metric on Ω; and the second one is introduced by Fefferman on a line bundle over $b\Omega$.

Let us first consider complete Kähler metrics on an open domain Ω. Suppose that

$$\omega_u = i\partial\bar{\partial}[\log(-\frac{1}{u})]$$

is a Kähler form on Ω. Such a Kähler form ω_u corresponds to a Kähler metric

$$g_u(X, Y) = \omega_u(X, JY) = i\partial\bar{\partial}[\log(-\frac{1}{u})](X, JY), \tag{2.1}$$

where J is the complex structure of Ω.

Secondly, Fefferman and his school considered a class of Lorentz metrics on canonical bundle on K^* mentioned above.

We will use an extrinsic way to define such metrics, along the line described in a new book [DT, p150]. Suppose that $\Lambda_{(n,0)}(V^{2n})$ be the canonical line bundle of open domain V^{2n}. Clearly, $\mathcal{L}_{V^{2n}} = \Lambda_{(n,0)}(V^{2n})$ is a complex manifold of complex dimension $(n+1)$.

When ξ is a cross-section of $\mathcal{L}_{V^{2n}}$ over V^{2n}, the norm $|\xi|_{g_u}$ induced by g_u is well-defined. We further define

$$H_u(z, \xi) = |\xi|_{g_u}^{\frac{2}{n+1}} u(z)$$

There is an $(1,1)$-form defined on $\mathcal{L}_{V^{2n}}$ given by $i\partial\bar{\partial}H_u$.

Similarly, there is a Hermitian form

$$G_u(\tilde{X}, \tilde{Y}) = i\partial\bar{\partial}H_u(\tilde{X}, \tilde{J}\tilde{Y}), \tag{2.2}$$

where $\tilde{J}$ is the complex structure of line bundle $\mathcal{L}_{V^{2n}}$. The Hermitian form G_u is *not* necessarily positive definite on the complex manifold $\mathcal{L}_{V^{2n}}$.

We now consider a subset

$$\Sigma^{2n} = \{(z, \xi) \in \mathcal{L}_{V^{2n}} \quad | \quad z \in b\Omega, |\xi| = 1\} \tag{2.3}$$

where Ω is an open, bounded and strictly pseudo-convex domain in V^{2n}.

Finally, when $i\partial\bar{\partial}u > 0$ on $M = b\Omega$, we consider

$$g_u^+ = G_u|_{\Sigma^{2n}}. \tag{2.4}$$

It was shown that g_u^+ is a Lorentz metric on Σ^{2n}. Clearly, Σ^{2n} is diffeomorphic to the unit circle bundle K^* mentioned above.

We remark that the function $u = 0$ *vanishes* on M^{2n-1}. The leading term of the metric g_u^+ is

$$i\partial\bar{\partial}u.$$

In [FH], Fefferman and Hirachi studied the so-called Q-curvature of CR-manifold M^3:

$$Q_{\theta_u}^{CR} = \frac{4}{3}(\Delta_b R - 2Im\nabla^\alpha\nabla^\beta A_{\alpha\beta}),$$

where R is the Tanaka-Webster scalar curvature, A is the torsion, Δ_b is the sub-Laplacian computed in terms of the contact 1-form θ_u and $\theta_u(\xi) = du(J\xi)$ for all $\xi \in T(M)$.

For higher dimensional manifolds, the Q-curvatures of higher order have been studied in [FH] and [GG].

The notations above will be used in the next two sub-sections.

2.b. Relations between the Fefferman's Lorentz metric and the Cheng-Yau's Kähler-Einstein metric.

In this sub-section, we illustrate a strategy to obtain the existence of Q-flat metrics on real hypersurfaces in $\mathbb{C}^n$.

Let us now recall a result obtained by Fefferman and his school.

Proposition 2.1. *([FG1, Chapter III]) Let $\Omega \subset V^{2n}$, $M = b\Omega \subset \mathbb{C}^n$, $u = \hat{u}e^\eta$ and $\{g_u, g_u^+\}$ be as above. If the Cheng-Yau metric g_u is a complete Kähler-Einstein on Ω, then the Lorentz metric g_u^+ is Einstein on Σ^{2n}.*

Here is a direct application of Propositions 2.0-2.1.

Corollary 2.2. *([FH], [GG]) Let $\Omega \subset \mathbb{C}^{2n}$ be an open strictly pseudo-convex domain with compact closure and let $M^{2n-1} = b\Omega$ be its boundary. Then M admits a metric of zero CR Q-curvature.*

Proposition 2.1 and Corollary 2.2 were stated for strictly pseudo-convex and bounded domain Ω in $\mathbb{C}^n$. We would like to extend these results to any strictly pseudo-convex and bounded domain Ω in a Stein manifold V^{2n}.

2.c. Compact smooth real hypersurfaces in a Stein manifold.

Our goal of this section is to verify the following theorem.

Proposition 2.3. *Let Ω be a bounded, open and strictly pseudo-convex domain with a smooth boundary in a Stein manifold V^{2n}. If the metric g_u above is a complete Kähler-Einstein metric on Ω, then g_u induces a metric $\tilde{g}_u^\infty$ on $M = b\Omega$ with zero CR Q-curvature.*

Proof. Since V^{2n} is Stein, we may assume that $V^{2n} \subset \mathbb{C}^m$ is a complete submanifold of $\mathbb{C}^m$, for sufficiently large m. Let $\hat{g}$ be induced metric on $\Omega \subset V^{2n} \subset \mathbb{C}^m$. For each local holomorphic coordinate system $\{(z_1, ..., z_n)\}$ of Ω, the Ricci tensor $\hat{Ric}$ of $\hat{g}$ is given by

$$\hat{Ric} = -i\partial\bar{\partial} \log[\det \hat{g}_{i\bar{j}}].$$

It is clear that $\hat{Ric}$ is well-defined and independent of the choice of local holomorphic coordinate system $\{(z_1, ..., z_n)\}$. Moreover, $\hat{Ric}$ is a closed $(1,1)$-form on Ω. In what follows, we first would like to solve Poincare-Lelong equation $i\partial\bar{\partial} f = \hat{Ric}$.

For this purpose, we recall a theorem of Dolbeault:

$$H^{(1,1)}(\Omega) = H^{(0,1)}(\Omega, \mathcal{O}|_\Omega)$$

where $\mathcal{O}|_\Omega$ is the bundle of holomorphic $(1,0)$-forms.

Since Ω is strictly pseudo-convex and bounded domain in a Stein manifold V^{2n}, by a theorem of Andreotti and Vesentini [AV], we have

$$H^{(0,1)}(\Omega, \mathcal{O}|_\Omega) = 0.$$

In fact, Proposition A.4 of [CaWS, p218] is also applicable for (0, q)-forms with values in $\mathcal{O}|_\Omega$. Thus, $H^{(1,1)}(\Omega) = H^{(0,1)}(\Omega, \mathcal{O}|_\Omega) = 0$. Professor Siu also handled similar formula with values in a vector bundle E, although the weighted functions were not discussed there (cf. [Siu, Chapters 2-3]). Hence, the first Chern class $c_1(\mathcal{O}|_\Omega) = 0$. Recall that, by Chern-Weil theory, the co-homology class $c_1(\mathcal{O}|_\Omega)$ is independent of the choices of affine connections, (cf. [Mi]). Therefore, $c_1(\mathcal{O}|_\Omega) = 0$ implies that the Chern-Weil form $\hat{Ric}$ is d-exact on Ω.

Therefore, we have $\hat{Ric} = d\beta$ for some 1-form β. Let us consider the decomposition of $\beta = \beta^{(1,0)} + \beta^{(0,1)}$, where $\beta^{(0,1)}$ is the $(0,1)$-component of β. If $\hat{Ric} = d\beta$ and if $\beta = \beta^{(0,1)} + \beta^{(1,0)}$, then $\bar\partial\beta^{(0,1)} = 0$, where we used the fact that $\hat{Ric}$ is an (1,1)-form. Choosing f with $\bar\partial f = i\beta^{(0,1)}$, we get a solution $i\partial\bar\partial f = \hat{Ric}$.

Recall that $\hat{Ric}$ is real valued. Replacing f by $Re\{f\}$ if needed, we conclude that the Poincare-Lelong equation

$$i\partial\bar\partial f = \hat{Ric} = -i\partial\bar\partial \log[\det \hat{g}_{i\bar{j}}].$$

has a smooth real-valued solution f on $\Omega \cup b\Omega$. Such a solution f is unique up to adding a pluri-subharmonic function. If we require that f has the smallest $L^2(\Omega)$-norm, then such a solution is unique, see Chapters 4-5 of [CS]. Such a solution f is called a Ricci potential of $\hat{g}$.

Following Fefferman [F2], we consider

$$\hat{J}(u) = (-1)^n e^{-f} \frac{1}{\det \hat{g}_{i\bar{j}}} \det \begin{pmatrix} u & u_{\bar{j}} \\ u_i & u_{i\bar{j}} \end{pmatrix} \tag{2.5}$$

where f is the Ricci potential of $\hat{g}$ as above, $u_i = \frac{\partial u}{\partial z_j}$, $u_{i\bar{j}} = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}$ and $\{z_1, ..., z_n\}$ is a local holomorphic frame.

When $\Omega \subset \mathbb{C}^n$, we choose the standard coordinate system. Thus, in this case, $\det \hat{g}_{i\bar{j}} = 1$ and we can choose $f = 0$. Therefore, our definition coincides with Fefferman's definition for the case of $\Omega \subset \mathbb{C}^n$, see [F2] and [CY].

A calculation similar to [CY, p508] further shows that the metric g_u is Kähler-Einstein of negative curvature $-(n+1)$ if

$$\frac{\det \varphi_{i\bar{j}}}{\det \hat{g}_{i\bar{j}}} = e^f e^{(n+1)\varphi} \tag{2.6}$$

holds, where $\varphi = \log(-\frac{1}{u})$.

A further calculation shows that the above equation holds if and only if

$$\hat{J}(u)|_z \equiv 1 \tag{2.7}$$

holds for all $z \in \Omega$.

It is known that if $\hat{J}(u)|_z \equiv 1$ in Ω, then $M = b\Omega$ has zero CR Q-curvature, see [FH, Chapter 3]. This completes the proof of Proposition 2.3. $\square$

Corollary 2.4. *Suppose that $\Omega \subset V^{2n}$ be a bounded, open and strictly pseudo-convex domain with smooth boundary in a Stein manifold V^{2n} with $n \geq 2$. Then its boundary $M^{2n-1} = b\Omega$ admits a metric of zero Q-curvature.*

Proof. By Proposition 2.3, it remains to verify that there is a complete Kähler-Einstein metric g_u on Ω. The existence of such a complete Kähler-Einstein metric g_u is provided by Mok-Yau in [MY, p52]. In fact, Mok and Yau found desired solutions $u = e^{\eta}\hat{u}$ and $\varphi = \log(-\frac{1}{u})$ satisfying $\frac{\det \varphi_{i\bar{j}}}{\det \hat{g}_{i\bar{j}}} = e^f e^{(n+1)\varphi}$. $\square$

3. Existence of Pseudo-Einstein metrics on CR-hypersurfaces of real dimension ≥ 5

In this section, we discuss the existence of pseudo-Einstein metrics on CR-hypersurfaces of real dimension ≥ 5. A metric g defined on a CR-manifold M^{2n-1} is said to be *pseudo-Einstein (or partially Einstein)* if its Ricci tensor satisfies

$$Ric_g(X, Y)|_z = \lambda g(X, Y)|_z \tag{3.0}$$

for some constant $\lambda = \lambda(z)$ and for all real vectors $\{X, Y\}$ in the CR-distribution $ker(\theta)|_z$, where θ is the contact form of M^{2n-1}.

One of our new contributions in this section is to use the $\bar{\partial}_b$-theory to solve boundary version of Poincaré-Lelong equation related to the *partially Einstein equation*, see Proposition 3.4 and Corollary 3.5 below.

When $\dim_{\mathbb{R}}[M^{2n-1}] = 3$, any metric g on M^3 is *pseudo-Einstein (i.e., partially Einstein)*. Therefore, we only consider the case of $\dim_{\mathbb{R}}[M^{2n-1}] \geq 5$.

We emphasize that a pseudo-Einstein metric g on M^{2n-1} is *not* necessarily Einstein. The pseudo-Einstein condition puts *no* restriction on its Ricci curvature in the directions which are transversal to CR-distribution. It might happen that

$$Ric_g(Z, Y) \neq \lambda g(Z, Y)$$

for some transversal vector $Z \perp ker(\theta)$.

In [L2], Lee already showed that, if a compact strongly pseudo-convex CR-manifold M^{2n-1} admits a closed, nowhere vanishing $(n, 0)$-form, then M^{2n-1} admits a pseudo-Einstein metric. In particular, if $M = b\Omega$ and $\Omega \subset \mathbb{C}^n$, then M admits a pseudo-Einstein structure.

We make extra observations to extend Lee's result to the case of $\Omega \subset V^{2n}$ for any Stein manifold V^{2n}. The new ingredient of our approach will use the fact that the Chern curvature forms Θ are type of $(1, 1)$ for Lorentz-Kähler metrics.

In addition, we will use Kohn's $\bar{\partial}_b$-theory to solve the boundary version of Poincare-Lelong equation

$$i\partial_b\bar{\partial}_b f = \Theta \tag{3.1}$$

for any $\bar{\partial}_b$-closed $(1, 1)$-form Θ.

The equation (3.1) above is related to the existence of pseudo-Einstein metrics, as described in [L2, p173]. Such an equation was previously studied in [CaWS] for other purposes.

It is well-known that, for any function u, one has

$$(d^c u)(\xi) = (du)(J\xi) \text{ and } dd^c u = i\partial\bar{\partial}u.$$

We begin with an elementary observation.

Lemma 3.1. *Let $\hat{u}$ be a defining function of $M = b\Omega$. Suppose that $\Omega \subset V^{2n}$ is a strictly pseudoconvex bounded domain in a Stein manifold. Then*

(1) There is another defining function $u = e^{\varphi}\hat{u}$ such that u is a strictly pluri-subharmonic in a neighborhood of $M = b\Omega$, i.e., $i\partial\bar{\partial}u > 0$.

(2) When $i\partial\bar{\partial}u > 0$ and $\theta_u = d^c u$, then $i\partial\bar{\partial}u$ gives rise to a Kähler metric g_u in a neighborhood of M.

(3) If $u = e^{\varphi}\hat{u}$, $\theta = d^c u$ and $\hat{\theta} = d^c\hat{u}$, then one has

$$\theta = e^{\varphi}\hat{\theta} \text{ on } M.$$

Proof. Assertion (1) was stated in Theorem 3.4.4 of [CS, p45-46].

The verification of Assertions (2)-(3) is straightforward. □

Proposition 3.2. *Let Ω be a bounded, strictly pseudo-convex domain with a smooth boundary $M = b\Omega$ in a Stein manifold V^{2n}, let $\mathcal{O}$ be the holomorphic $(1,0)$-form bundle of V^{2n}, and let K^* be the canonical line bundle of V^{2n}. Suppose that $\dim_{\mathbb{R}}[V^{2n}] = 2n \geq 6$. Then the following is true.*

(1) The first Chern class of $\mathcal{O}|_\Omega$ is equal to zero, i.e., $c_1(\mathcal{O}|_\Omega) = 0$; Moreover, the first Chern class of canonical line $c_1(K^*|_M) = 0$;

(2) The Ricci curvature form Ric_g of any metric $g = d\theta$ on $\mathcal{O}|_\Omega$ is a d-exact $(1,1)$-form on M. Furthermore, $Ric(\xi, \bar{\xi})$ is a real number for all $\xi \in T^{(1,0)}(M)$.

Proof. (1) We will use curved version of Kohn-Morrey formula to verify that

$$c_1(\mathcal{O}|_\Omega) = 0. \tag{3.2}$$

Recall that the closure $\bar{\Omega}$ of Ω is compact. Since V^{2n} is a Stein manifold, there is a strictly pluri-subharmonic function ϕ_0. Let $\phi = \lambda\phi_0$ for sufficiently large $\lambda > 0$. Using Bochner-Hörmander-Kohn-Morrey formula, we obtain

$$H^{(p,q)}(\Omega) = 0, \tag{3.3}$$

for all $0 < q < n$, (cf. Proposition A.4 of [CaWS, p218]).

It is well-known that, for $\dim_{\mathbb{C}}(\Omega) = n > 2$

$$H^1(\Omega, \mathcal{O}|_\Omega) = H^{(1,1)}(\Omega) = 0. \tag{3.4}$$

It follows that the first Chern class of $\mathcal{O}|_\Omega$ is zero.

Choose a Kähler metric $\hat{g}$ on Ω. Then the Ricci curvature form $\hat{\Theta}$ is a d-exact $(1,1)$-form.

The classical Kohn-Rossi theory states that any $\bar{\partial}_b$-closed $(1,0)$-form on $M = b\Omega$ can be extend to a unique holomorphic $(1,0)$-form on the whole Ω. Thus,

$$H^{(1,1)}(M) = 0, \tag{3.5}$$

see [KoR].

It is also known that $c_1(\mathcal{O}|_M) = c_1(K^*|_M) = 0$.

(2) Let g_u be the Kähler metric associated with the Kähler form $i\partial\bar{\partial}u$. The corresponding first Chern curvature form Θ_u of the Kähler metric g_u is a closed $(1,1)$-form in a neighborhood of M in V^{2n}.

The classical Chern-Weil theory implies that the cohomology class of the first Chern curvature form $\Theta|_M$ is independent of the choice of the choice of affine connections on M.

In fact, if $\theta_u = d^c u$ and $i\partial\bar{\partial}u > 0$, then $d\theta_u = dd^c u = i\partial\bar{\partial}u > 0$ gives rise a Kähler metric in a neighborhood of M. For any other $\tilde{\theta} = e^{2\varphi}\theta$, the Ricci curvature form corresponding to $\tilde{\theta}$ remains to be of type $(1,1)$, see Lemma 2.4 of [L2]. $\square$

We now recall that a result of Lee [L2].

Proposition 3.3. *([L2, Lemma 6.1, p173-174]) Let $M = b\Omega$ and $\Omega \subset V^{2n}$ be as in Main Theorem. Suppose that $\tilde{\theta} = e^{2u}\hat{\theta}$ and $\hat{R}ic$ is the Ricci curvature form corresponding to $\hat{\theta}$. Then $\tilde{\theta}$ is pseudo-Einstein if and only if there is a real solution u satisfying*

$$i\partial_b\bar{\partial}_b u = \hat{R}ic$$

Proof. By (6.3) of [L2], the trace-less part of $\tilde{Ric}$ is zero if there is φ satisfying

$$(n+1)i\partial_b\bar{\partial}_b\varphi = \hat{Ric}$$

Since $\hat{Ric}$ is a real valued d-exact real-valued $(1,1)$-form by Proposition 3.2 above, we can choose φ to be real-values as well. (Otherwise, let $v = \frac{1}{2}(\varphi+\bar{\varphi})$ instead). $\square$

Proposition 3.4. *Let $M = b\Omega$ and $\Omega \subset V^{2n}$ be as in Main Theorem. Suppose that the Ricci curvature form $\hat{Ric}$ form is a d-exact $(1,1)$-form for the contact 1-form $\hat{\theta}$. Then there always a real-valued function u satisfying*

$$i\partial_b\bar{\partial}_b u = \hat{Ric} \tag{3.6}$$

Proof. Choose σ such that

$$d\sigma = \hat{Ric}. \tag{3.7}$$

Let $\sigma = \sigma^{(0,1)} + \sigma^{(1,0)} + \lambda\theta$, where $\sigma^{(0,1)}$ is $(0,1)$-component of σ. Since $\hat{Ric}$ is of type $(1,1)$, by (3.7) we have

$$\bar{\partial}_b\sigma^{(0,1)} = 0. \tag{3.8}$$

Because $\dim_{\mathbb{C}}(\Omega) > 2$, by a Theorem of Kohn that there is complex-valued function f with

$$i\bar{\partial}_b f = \sigma^{(0,1)}, \tag{3.9}$$

see [CS, Ch9].

It follows that

$$i\partial_b\bar{\partial}_b f = \partial\sigma^{(0,1)} = (d\sigma)_b = (\hat{Ric})_b. \tag{3.10}$$

Since $(\hat{Ric})_b$ is real-valued (1,1)-form, choosing $u = Re\{f\}$, we are done. $\square$

We now summarize our result of this section.

Corollary 3.5. *Suppose that $\Omega \subset V^{2n}$ be a compact strictly pseudo-convex domain with smooth boundary in a Stein manifold V^{2n}. Then its boundary $M^{2n-1} = b\Omega$ admits an intrinsic pseudo-Einstein (i.e., partially Einstein) metric.*

Proof. This is a direct consequence of Lemma 3.1 and Propositions 3.2-3.4. $\square$

4. Estimates for CR Paneitz operators on M^3

In the remaining of this paper, we study the so-called CR Paneitz operator

$$P_u f = \triangle_b^2 f + T^2 f + 4Im\nabla_\beta(A^{\alpha\beta}\nabla_\alpha f), \tag{4.1}$$

where $T = J\nabla u$ is the Reeb vector and A is the torsion tensor of the contact form θ_u.

When the torsion A vanishes, the formula (4.1) reduces to (1.2).

It remains to verify that CR Paneitz operator P_u is a closed operator.

If $\hat{\theta} = e^{\varphi}\theta_u$ on M^3 and $\hat{Q}$ is the corresponding CR Q-curvature of the metric associated with the contact form $\hat{\theta}$, then

$$e^{2\varphi}\hat{Q} = Q + P_u\varphi,$$

see (5.14) of [GG].

Our goal is to show the following result.

Proposition 4.1. *Let $\Omega \subset V^4$ be an open strictly pseudo-convex domain with compact closure in a Stein manifold V^4 and let $M^3 = b\Omega$ be its boundary. Suppose that g_u is the Cheng-Yau Einstein metric on Ω and $\theta_u(.) = du(J.)$ is the corresponding contact 1-form on M^3. Then the Paneitz operator P_u is closed:*

$$\int_{M^3} |P_u f|^2 \geq c \int_{M^3} |f|^2, \tag{4.2}$$

for any real valued function $f \perp ker P_u$, where $c > 0$ is a constant independent of f.

Remark 4.2: The constant c in Proposition 4.1 depends mostly on the Tanaka-Webster curvature R and pseudo-hermitian torsion A_{11} of (M^3, J, θ_u) respectively. In fact, the following holds:

$$\int_M 2(Pf)f\theta_u \wedge d\theta_u = \int_M [3(\triangle_b f)^2 - |Hess_b f|^2 - R|\nabla_b f|^2 - 6Im\{A_{\overline{11}}f_1 f_1\}]\theta_u \wedge d\theta_u$$

where ∇_b and $Hess_b^2$ denotes the sub-gradient and sub-Hessian with respect to (J, θ_u) respectively, see [CC].

For the proof of Proposition 4.1, we need some notations.

In what follows, we let $\theta = \theta_u$ be the given contact form. The vector T is the characteristic vector in $T(M)$ such that $\theta(T) = 1$, $(d\theta)(T, .) = 0$.

An (1, 0)-form $\theta^1 \in \Lambda_{(1,0)}(M^3)$ is called *admissible* if

$$\theta^1(T) = 0, d\theta = ih_{1,\bar{1}}\theta^1 \wedge \theta^{\bar{1}}$$

for some hermitian metric function $h_{1,\bar{1}}$.

It is known that

$$\Delta_b f = -f_\alpha{}^\alpha - f_{\bar{\alpha}}{}^{\bar{\alpha}}$$

and

$$\Box_b f = 2(\bar{\partial}_b^* \bar{\partial}_b + \bar{\partial}_b \bar{\partial}_b^*) f = (\Delta_b + iT) f = -2 f_{\bar{\alpha}}{}^{\bar{\alpha}}.$$

Inspired by proof of Proposition 3.4 of [L2], we will express the CR Paneitz operator P as a product of several closed operators.

We first consider

$$\mathcal{L} f = d_b^c f + (\Delta_b f)\theta, \tag{4.3}$$

where θ is the contact 1-form described above.

Lemma 4.2. *Let $M^3 = b\Omega$, θ, A and $\mathcal{L}$ be as above. Suppose that $\Omega \subset V^4$ is a strictly pseudo-convex domain in a Stein manifold V^4 and that Ω has compact closure. Then $\mathcal{L}$ is a closed operator.*

Moreover, one has

$$d[\mathcal{L} f] = 2(f_{\bar{1}}{}^{\bar{1}}{}_1 + iA_{11} f^1)\theta \wedge \theta^1 + 2(f_1{}^1{}_{\bar{1}} - iA_{\bar{1}\bar{1}} f^{\bar{1}})\theta \wedge \theta^{\bar{1}}.$$

Proof. By Theorem 9.4.2 of [CS], both d_b^c and $\triangle_b$ are closed operators for strictly pseudo-convex compact CR-hypersurfaces. Notice that $d_b^c f \in [\Lambda_{(1,0)}(M^3) \oplus \Lambda_{(0,1)}(M^3)]$ is always orthogonal to the 1-form $(\Delta_b f)\theta$. Hence, $\mathcal{L}$ is a closed operator.

We will use the proof of Proposition 3.4 of [L2].

The $\theta^1 \wedge \theta^{\bar 1}$-component of $d[\mathcal{L}f]$ is

$$i[f_{1\bar 1} + f_{\bar 1 1} - (f_1^1 + f_{\bar 1}^{\bar 1})h_{1\bar 1}]\theta^1 \wedge \theta^{\bar 1} = 0.$$

On the other hand, the $\theta \wedge \theta^1$-component of $d[\mathcal{L}f]$ is

$$[{f_1}^1{}_1 + {f_{\bar 1}}^{\bar 1}{}_1 - if_{1,0} + iA_{11}f^1]\theta \wedge \theta^1. \tag{4.4}$$

It is known (cf. [L2, Section 2]) that

$$-{f_1}^1{}_1 + {f_{\bar 1}}^{\bar 1}{}_1 + if_{1,0} + iA_{11}f^1 = 0. \tag{4.5}$$

It follows from (4.4) and (4.5) that the $\theta \wedge \theta^1$-component of $d[\mathcal{L}f]$ is equal to

$$2({f_{\bar 1}}^{\bar 1}{}_1 + iA_{11}f^1)\theta \wedge \theta^1. \tag{4.6}$$

For the same reason, the $\theta \wedge \theta^{\bar 1}$-component of $d[\mathcal{L}f]$ is equal to

$$2({f_1}^1{}_{\bar 1} - iA_{\bar 1\bar 1}f^{\bar 1})\theta \wedge \theta^{\bar 1}. \tag{4.7}$$

This completes the proof. $\square$

Proof of Proposition 4.1. We now consider the composition of operators:

$$\tilde{P}f = \partial_b^*[\big(d(\mathcal{L}f)\big)\lfloor_T]. \tag{4.8}$$

It follows from that

$$\big(d(Lf)\big)\lfloor_T = 2({f_{\bar 1}}^{\bar 1}{}_1 + iA_{11}f^1)\theta^1 + 2({f_1}^1{}_{\bar 1} - iA_{\bar 1\bar 1}f^{\bar 1})\theta^{\bar 1}. \tag{4.9}$$

We observe that ∂_b^* acts on $\Lambda_{(1,0)}(M^3)$ trivially. For real valued function f, we further consider

$$Re[\tilde{P} \circ f] = Re[\overline{\Box}_b \Box_b f] + 4Im(A_{\bar 1 1}f_1)_1, \tag{4.10}$$

where $Re\{z\}$ is the real part of complex number of z.

Therefore, it follows from (4.8)-(4.10) that, for real valued function f, we have

$$Re[\tilde{P}f] = \triangle_b^2 f + T^2 f + 4Im(A_{\bar{1}\bar{1}} f_1)_1 = Pf. \tag{4.11}$$

Thus, the CR Paneitz operator P satisfies

$$Pf = Re[\tilde{P}f], \tag{4.12}$$

where

$$\tilde{P} = \partial_b^*[\big(d(\mathcal{L}f)\big)\lfloor_T].$$

A composition of closed operators remains to be a closed operator.

If $M^3 = b\Omega$ is a compact strictly pseudo-convex hypersurface in a Stein manifold V^4, then $\{\bar{\partial}_b, d, \partial_b^*, \mathcal{L}\}$ are closed operators, by Kohn's $\bar{\partial}_b$-theory (cf. [CS, Theorem 9.4.2, p231]). Theorem 9.4.2 of [CS] was stated for $\Omega \subset \mathbb{C}^2$, but its proof is applicable to Ω in all Stein manifolds V^4 including $\mathbb{C}^2$. It is clear that the operator Re is a closed operator. Therefore, $P = Re\tilde{P}$ is a closed operator as well. $\square$

Proof of Main Theorem. Main Theorem now follows from Corollary 2.4, Corollary 3.5 and Proposition 4.1. $\square$

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